Gravitational waves from a neutron star merger

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Abstract: In this work we present the equations that describe the structure of neutron stars, as well as different models that describe the properties of the neutron star matter, using Skyrme nuclear interactions. From the data obtained in the gravitational waves detected in recent years we determine which models are still valid to describe the neutron star matter and which are not.

I. INTRODUCTION

For the first time in history, in 2017 gravitational waves and electromagnetic signals from the same source were detected simultaneously thanks to the collaboration of the LIGO and Virgo detectors. Specifically, the gravitational waves created by the merger of two neutron stars (NS). Two inspiralling NS eventually merged into a single mass, releasing energy in various forms, including gravitational waves, matter and light. The gravitational waves provided information on each star's properties, such as its mass and its "tidal deformability" – the stiffness of a star in response to the stresses caused by its companion's gravitational field.

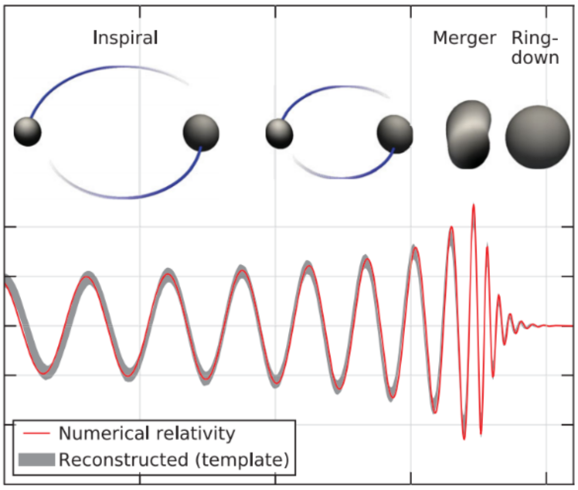

FIG. 1: Binary black hole merger [1], corresponding to the gravitational waves detected on 2015, GW150914.

Using parameters such as tidal deformability can significantly restrict the family of allowed equations of state (EoS) models that describe dense nuclear matter, since due to the complexity of dense nuclear matter and the dearth of experimental inputs, many EoS are possible (see e.g. Ref [2]).

The second section will show the equations that describe the structure of an NS as well as those that allow to calculate the tidal deformability and the k_2, the second number Love. We will also deduce how to obtain the neutron star matter equation of state. In the next section we will introduce the Skyrme model, which is widely used in the literature to describe nuclear interaction and which will help us to obtain the equation of state for a neutron star. In the results section we will discuss how the state equations behave for different Skyrme models as well as the mass-radius ratio, and the tidal deformability and the k_2 with respect to mass. We will compare the values obtained theoretically by these models with those obtained observationally from the 2017 detection of gravitational waves emitted in the merger of two neutron stars, GW170817. Finally, we will comment on the conclusions of the work.

II. THEORY

A. Structure equations (TOV) and tidal deformability

In neutron stars, there are two opposite forces acting on the star, one of them is gravitation and the second one arises from degeneracy pressure and nuclear interactions. Since NS are very compact, one has to take into account effects from general relativity, like the curvature of spacetime. We then must describe a compact star by using Einstein's equation

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{1}$$

For an isotropic, static, ideal fluid sphere in hydrostatic equilibrium, one arrives at the Tolman-Oppenheimer-Volkoff (TOV) equations [3]:

$$\frac{dp}{dr} = -\frac{G\epsilon(r)m(r)}{c^2 r^2}\left[1+\frac{p(r)}{\epsilon(r)}\right] \times\left[1+\frac{4\pi r^3 p(r)}{m(r)c^2}\right]\left[1-\frac{2Gm(r)}{c^2 r}\right]^{-1} \tag{2}$$

$$\frac{dm}{dr} = \frac{\epsilon(r)}{c^2} 4\pi r^2 \tag{3}$$

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In the pressure equation, eq. (2), the first term corresponds to the newtonian version of the equation, followed by three correction factors. The last term contains the factor $\frac{2GM}{c^2 r}$ which determines whether the relativistic effects must be considered or not. The corresponding critical radius $R = \frac{2GM}{c^2}$ is the so-called Schwarzschild radius [4]. Thus, equations (2) and (3) are the equations that determine the structure of the star.

The tidal deformability measures the quadrupole deformation of a NS as a response to the perturbing tidal field of the companion in a binary system. For each star we can calculate its tidal deformability as [5]

$$\Lambda = \frac{2}{3} k_2 \left[\left(\frac{c^2}{G} \right) \left(\frac{R}{M} \right) \right]^5 \tag{4}$$

where k_2 is the second Love number and R is the stellar radius. Both R and k_2 are fixed for a given stellar mass M by the EoS for NS matter, with $k_2 \approx 0.05 - 0.15$ for realistic neutron stars [6]. The Love number is given by

$$\begin{aligned} k_2 &= \frac{8\chi^5}{5}(1-2\chi)^2[2+2\chi(y-1)-y] \\ &\times \{2\chi[6-3y+3\chi(5y-8)] \\ &+ 4\chi^3[13-11y+\chi(3y-2)+2\chi^2(1+y)] \\ &+ 3(1-2\chi)^2[2-y+2\chi(y-1)]\ln(1-2\chi)\}^{-1}, \end{aligned} \tag{5}$$

where

$$\chi = \frac{GM}{Rc^2} \tag{6}$$

is the compactness of the star and

$$y = \frac{R\beta(R)}{H(R)}. \tag{7}$$

One can obtain the values of $\beta(R)$ and $H(R)$ by solving the following differential equations [7]:

$$\frac{dH(r)}{dr} = \beta(r) \tag{8}$$

$$\begin{aligned} \frac{d\beta(r)}{dr} &= \frac{2G}{c^2}\left(1 - \frac{2Gm(r)}{rc^2}\right)^{-1} H(r) \\ &\times \left\{ -2\pi \left[5\epsilon + 9p + \frac{d\epsilon}{dp}(\epsilon + p) \right] + \frac{3c^2}{r^2 G} \right. \\ &\left. + \frac{2G}{c^2}\left(1 - \frac{2Gm(r)}{rc^2}\right)^{-1} \left(\frac{m(r)}{r^2} + 4\pi r p \right)^2 \right\} \\ &+ \frac{2\beta(r)}{r}\left(1 - \frac{2Gm(r)}{rc^2}\right)^{-1} \\ &\times \left\{ -1 + \frac{Gm(r)}{rc^2} + \frac{2\pi r^2 G}{c^2}(\epsilon - p) \right\}. \end{aligned} \tag{9}$$

In order to solve all these equations, we need to know the relation $p(\epsilon)$, that is, the EoS. Once the EoS of the NS matter is known, the TOV equations can be integrated from the origin, with initial conditions $m(0) = 0$ and an arbitrary value for the central pressure $p(0) = p_c$, until the pressure vanishes. The point $r = R$ where $p(R) = 0$ defines the radius R and the total mass $M = m(R)$ of the star. By repeating the calculation for different central pressures, the mass-radius relation of NS is obtained.

B. Equation of state of neutron star matter

We will describe the core of a NS as composed of a beta-stable uniform liquid of neutrons, protons and electrons (‘npe’ matter). It is not possible to have only neutrons because a neutron in free space decays into a proton. Therefore, there must be some fraction of protons and electrons, in addition to neutrons, to ensure beta-equilibrium through the reactions

$$\begin{aligned} n &\to p + e^- + \bar{\nu}_e \\ p + e^- &\to n + \nu_e. \end{aligned} \tag{10}$$

Chemical equilibrium implies that the chemical potentials of the particles must fulfill the condition

$$\mu_n = \mu_p + \mu_e. \tag{11}$$

One does not include the chemical potential of the neutrinos because it is assumed that the neutrinos leave the star as soon as they are produced, since the neutrino mean free path is larger than the typical radius of neutron stars. In addition to the beta-stable equilibrium condition, we must impose

$$\rho_p = \rho_e \tag{12}$$

because the NS is globally charge neutral. At the high densities of the core of the NS, the electrons can be described as an ultra-relativistic free Fermi gas at T=0. We consider the system at zero temperature since the density is so high that so is the Fermi energy, therefore all occupied states are below the Fermi level, just as they would be at T=0. For electrons, the ultra-relativistic expressions are:

$$\epsilon_e = \frac{3}{4}\hbar c \left(3\pi^2\right)^{1/3} \rho_e^{4/3} \tag{13}$$

$$p_e = \rho_e^2 \frac{\partial}{\partial \rho_e}\left(\frac{\epsilon_e}{\rho_e}\right) = \frac{1}{4}\hbar c \left(3\pi^2\right)^{1/3} \rho_e^{4/3} = \frac{\epsilon_e}{3} \tag{14}$$

$$\mu_e = \frac{\partial \epsilon_e}{\partial \rho_e} = \hbar c \left(3\pi^2 \rho_e\right)^{1/3} \tag{15}$$

Ultra-relativistic limit means $\hbar k_F c \gg m_e c^2$, so that $x_F \equiv \frac{k_F}{m_e} \gg 1$. We can rewrite the beta-equilibrium conditions (11) and (12) in terms of δ and ρ, where δ is the asymmetry parameter

$$\delta = \frac{\rho_n - \rho_p}{\rho}. \tag{16}$$

and ρ is the total baryon density

$$\rho = \rho_n + \rho_p. \tag{17}$$

The terms ρ_n and ρ_p correspond to the density of neutrons and protons, respectively.

$$\begin{aligned}\mu_n &= \frac{\partial \epsilon_n}{\partial \rho_n} = \frac{\partial \epsilon_n}{\partial \rho} + \frac{\partial \epsilon_n}{\partial \delta}\frac{\partial \delta}{\partial \rho_n} \\ \mu_p &= \frac{\partial \epsilon_p}{\partial \rho_p} = \frac{\partial \epsilon_p}{\partial \rho} + \frac{\partial \epsilon_p}{\partial \delta}\frac{\partial \delta}{\partial \rho_p}\end{aligned} \tag{18}$$

$$\frac{\partial \delta}{\partial \rho_n} = \frac{2\rho_p}{\rho^2}, \quad \frac{\partial \delta}{\partial \rho_p} = -\frac{2\rho_p}{\rho^2}. \tag{19}$$

$$\Rightarrow \mu_n - \mu_p = \frac{2}{\rho}\frac{\partial \epsilon_N}{\partial \delta} \tag{20}$$

Therefore, the beta-equation condition $\mu_n = \mu_p + \mu_e$ can be written as

$$\frac{2}{\rho}\frac{\partial \epsilon_N}{\partial \delta} = \hbar c \left(3\pi^2 \rho_e\right)^{1/3}. \tag{21}$$

As mentioned, charge neutrality requires $\rho_e = \rho_p \Rightarrow \rho_e = \left(\frac{1-\delta}{2}\right)\rho$. Considering the two conditions together:

$$\frac{2}{\rho}\frac{\partial \epsilon_N}{\partial \delta} = \hbar c \left(\frac{3\pi^2}{2}\right)^{1/3} \rho^{1/3} \left(1-\delta\right)^{1/3}. \tag{22}$$

For each density ρ, the solution of this equation gives the asymmetry δ that fulfills the beta-equilibrium condition and charge neutrality.

Now we just need to define the nuclear energy density, $\epsilon_N(\rho, \delta)$, which will vary depending on the model of nuclear interaction we use. We have used the Skyrme parametrized interactions, which are widely used in the literature and give realistic results of the properties of finite nuclei and nuclear matter.

III. NUCLEAR MODELS: SKYRME INTERACTION

There are different models that allow us to obtain the EoS. For any interaction, the saturation density of nuclear matter is $\rho_0 \sim 0.16$ fm^{-3}, and the energy per particle when $\rho = \rho_0$ is $E/A \sim -16.0$ MeV. These values have been obtained experimentally in laboratories on Earth. But high-density systems like those of neutron stars cannot be replicated on Earth, so different models differ mainly in their behaviour at high densities.

Skyrme models correspond to an effective parameterization of the contact interaction between nucleons [8]. We now introduce a relatively simple form of the Skyrme interaction as an example, although more general versions with more terms and parameters [2] have been used for the calculations:

$$\begin{aligned}\epsilon_N(\rho_n, \rho_p) &= \frac{\hbar^2 \tau_n}{2m} + \frac{\hbar^2 \tau_p}{2m} \\ &+ \frac{t_0}{2}\left[\left(1 + \frac{x_0}{2}\right)\rho^2 - \left(x_0 + \frac{1}{2}\right)\left(\rho_n^2 + \rho_p^2\right)\right] \\ &+ \frac{t_3}{12}\rho^\gamma \left[\rho^2 - \frac{1}{2}\left(\rho_n^2 + \rho_p^2\right)\right],\end{aligned} \tag{23}$$

where

$$\begin{aligned}\tau_n &= \frac{3}{5}\left(3\pi^2\right)^{2/3} \rho_n^{5/3}, \\ \tau_p &= \frac{3}{5}\left(3\pi^2\right)^{2/3} \rho_p^{5/3}\end{aligned} \tag{24}$$

denote the kinetic energy densities of neutrons and protons. In the nuclear matter, they are given by the usual expressions of the free Fermi gas. The coefficients t_0, t_3, x_0, γ are free parameters of the interaction. The term with t_0 describes the attractive two-particle nuclear interaction, and the term with t_3 describes the repulsive many-body interaction which is the dominant one at high nuclear densities.

The nuclear energy density can be rewritten in terms of δ and ρ:

$$\begin{aligned}\epsilon_N(\delta, \rho) &= \frac{\hbar^2}{2m}\frac{3}{10}\left(\frac{3\pi^2}{2}\right)^{2/3} \rho^{5/3}\left[(1+\delta)^{5/3} + (1-\delta)^{5/3}\right] \\ &+ \frac{t_0}{4}\left[\frac{3}{2} - \left(x_0 + \frac{1}{2}\right)\delta^2\right]\rho^2 + \frac{t_3}{48}\left(3 - \delta^2\right)\rho^{\gamma+2} \\ &+ \frac{3}{4}\hbar c \left(3\pi^2\right)^{1/3} \rho^{4/3}\left(\frac{1-\delta}{3}\right)^{4/3}.\end{aligned} \tag{25}$$

Now we can obtain the expression of the pressure $p(\delta, \rho)$, i.e., the equation of state (EoS).

$$\begin{aligned}p_N(\delta, \rho) &= \frac{\hbar^2}{2m}\frac{1}{5}\left(\frac{3\pi^2}{2}\right)^{2/3} \rho^{5/3}\left[(1+\delta)^{5/3} + (1-\delta)^{5/3}\right] \\ &+ \frac{t_0}{4}\left[\frac{3}{2} - \left(x_0 + \frac{1}{2}\right)\delta^2\right]\rho^2 + \frac{t_3}{48}\left(\gamma+1\right)\left(3 - \delta^2\right)\rho^{\gamma+2} \\ &+ \frac{1}{4}\hbar c \left(3\pi^2\right)^{1/3} \rho^{4/3}\left(\frac{1-\delta}{3}\right)^{4/3}\end{aligned} \tag{26}$$

Computing $\frac{\partial \epsilon_N(\rho,\delta)}{\partial \delta}$ with our Skyrme energy density we find that the beta-equilibrium condition becomes

$$\begin{aligned}&\frac{\hbar^2}{2m}\left(\frac{3\pi^2}{2}\right)^{2/3}\left[(1+\delta)^{2/3} - (1-\delta)^{2/3}\right]\rho^{2/3} \\ &\quad - t_0\left(x_0 + \frac{1}{2}\right)\rho\delta - \frac{t_3}{12}\rho^{\gamma+1}\delta \\ &\quad - \hbar c \left(\frac{3\pi^2}{2}\right)^{1/3} \rho^{1/3}(1-\delta)^{1/3} = 0.\end{aligned} \tag{27}$$

For each specific value of the nuclear density ρ, with this equation we can obtain the asymmetry parameter δ that

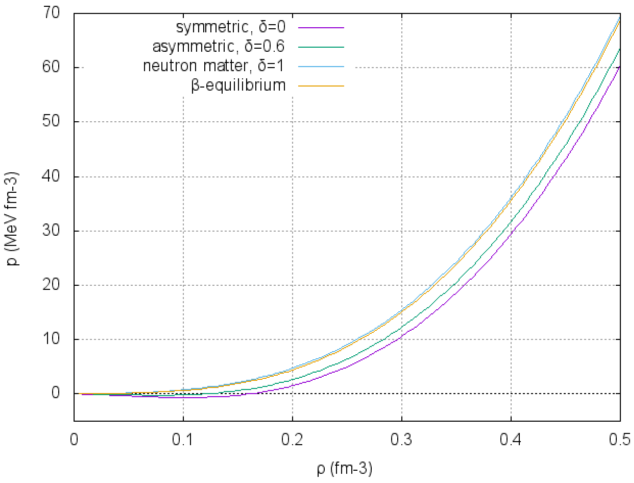

FIG. 2: $p(\rho)$ for different asymmetry parameters and β equilibrium for the SLy4 model.

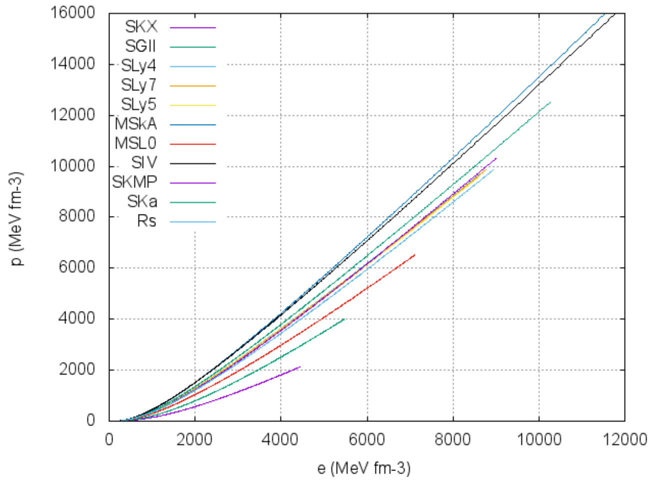

FIG. 3: $p(\epsilon)$ for different Skyrme models

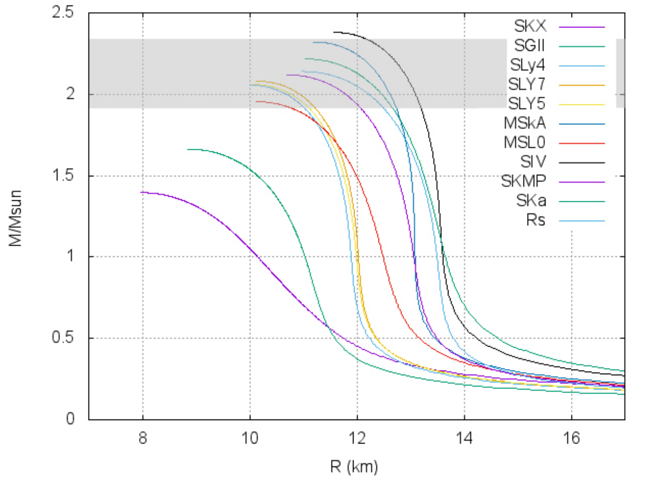

FIG. 4: NS mass-radius diagrams for different Skyrme models.

guarantees that the system is in beta-equilibrium. By replacing these δ and ρ in eqs (25) and (26) we have the energy density and pressure of the matter of the core of the NS. We can also compute the mass of the star in terms of the radius and the tidal deformability by solving the TOV equations.

In Fig. 2 we see how the EoS varies for different values of the asymmetry parameter. The curve for neutron matter is very similar to that of beta-equilibrium, as expected, since beta-stable matter is made up mostly of neutrons.

IV. RESULTS

We now study how the equations of state behave for different Skyrme models. We will then see which ones fit the experimental data. These models are highly used in the literature [2, 8, 9]. We have considered the following Skyrme interactions: SKX, SGII, SLy4, SLy7, SLy5, MSkA, MSL0, SIV, SKMP, SKa and Rs [2, 9].

We see in Fig. 3 that with a small variation in energy density we obtain smoother curves in some cases and stiffer ones in others, although they all evolve equally at low densities. The curves end at the last point where the equation of state is defined by a neutron star, although mathematically it could continue. Therefore, it indicates the point where the mass of the star is maximum, above which we would no longer have an NS.

As can be seen in Fig. 4, the different models find very different radii for the same mass, which have to do with the stiffness of the EoS shown in Fig. 3. In general, a larger maximum mass has a larger radius.

From the data obtained with the gravitational wave detection of 2017, GW170817, and its subsequent study, we can determine which models are still valid and which are not. We have two main constraints.

The total mass is one of the best established observables of neutron stars from many observational studies. Among them, there are the recent accurate observations of highly massive neutron stars, corresponding to $(1.928 \pm 0.017)M_\odot$ [10] and $(2.01 \pm 0.04)M_\odot$ [11] for the PSR $J1614$-2230 and PSR $J0348$+0432 pulsars, respectively. As a result, a great effort has been addressed to derive nuclear models able to generate EoSs that predict such massive objects. A very recent value for the observed largest NS mass is $2.14^{+0.20}_{-0.18}M_\odot$ at 95.4% credible level [12]. With these values, we place the minimum and maximum at $1.982 - 0.017$ and $2.14 + 0.20$ solar masses, respectively, as shown in Fig. 4 with the shaded area.

The LIGO/Virgo analysis of the GW170817 NS merger event constrained the tidal deformability at $M = 1.4M_\odot$, $\Lambda_{1.4} = 190^{+390}_{-120}$ [13], and so is shown in Fig. 5.

With these two constraints one can figure out which of the previous models are not valid to describe NS matter. The $M \sim 2M_\odot$ condition invalidates the SKX,

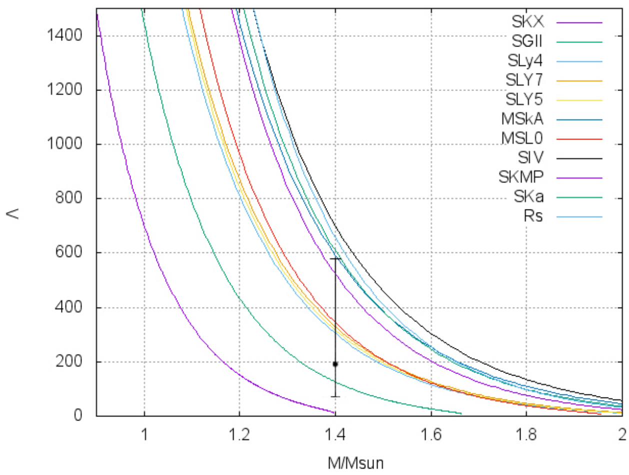

FIG. 5: Dimensionless tidal deformability as a function of the NS mass for different Skyrme models.

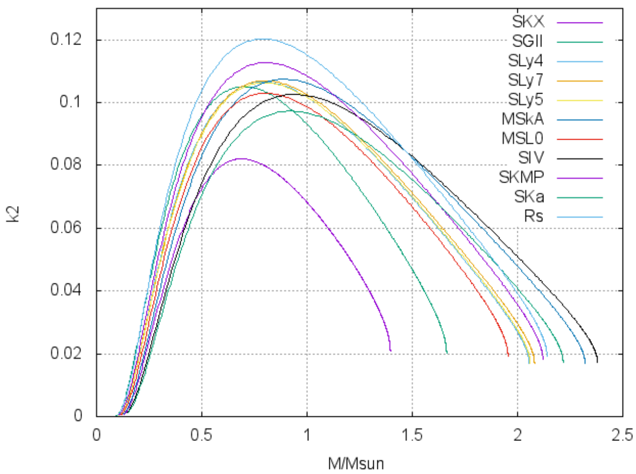

FIG. 6: Love number as a function of NS mass for different Skyrme models.

SGII and the SIV models. Fig. 6 also shows that each curve ends at the same maximum mass shown in Fig. 4. We see that the ratio k_2-mass, obtained by solving the equations (8) and (9), despite depending on the EoS, hardly varies between models except in cases where the maximum mass is significantly less than $2M_\odot$. In contrast, the tidal deformability, proportional to k_2 and R^5, does vary markedly for different models. The $\Lambda_{1.4} = 190^{+390}_{-120}$ constraint invalidates the SKX, MSkA, SKa, RS and SIV models. Therefore, the SLy4, SLY5, SLY7, MSL0 and SKMP are the only ones that satisfy both restrictions.

V. CONCLUSIONS

Given the inability to replicate high-density matter systems in a laboratory, the detection of gravitational waves signals emitted by these types of bodies is highly useful in studying them. In the case of neutron stars, having information about how they deform under a tidal force from another neutron star allows us to deduce properties of its matter, which is highly relevant to nuclear physicists.

Although with the data received so far we do not have enough to determine the equation of state of a neutron star, we can highly restrict the different theoretical models and approach to a more precise solution. Skyrme models used in this work have proved useful in order to study the different relations of neutron star matter. Properties such as the tidal deformability of neutron stars are highly useful for us to do this.

Acknowledgments

I would like to thank my advisor, Mario Centelles, for his help and patience. I would also like to thank my family who supported me, and to Luís Valero who checked my english writing.

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